SOME CONSTRUCTIONS OF BIHARMONIC MAPS AND CHEN'S CONJECTURE ON BIHARMONIC HYPERSURFACES

YE-LIN OU*

Dedicated to Professor J. C. Wood on his 60th birthday

Abstract

We give several construction methods and use them to produce many examples of proper biharmonic maps including biharmonic tori of any dimension in Euclidean spheres (Theorem 2.2, Corollaries 2.3, 2.4 and 2.6), biharmonic maps between spheres (Theorem 2.9) and into spheres (Theorem 2.10) via orthogonal multiplications and eigenmaps. We also study biharmonic graphs of maps, derive the equation for a function whose graph is a biharmonic hypersurface in a Euclidean space, and give an equivalent formulation of Chen's conjecture on biharmonic hypersurfaces by using the biahrmonic graph equation (Theorem 4.1) which paves a way for analytic study of the conjecture.

1. introduction

In this paper, all manifolds, maps, and tensor fields are assumed to be smooth if there is no otherwise statements.

A biharmonic map is a map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds that is a critical point of the bienergy functional

$$E^2(\varphi, \Omega) = \frac{1}{2}\int_{\Omega} |\tau(\varphi)|^2 \,\mathrm{d}x$$

for every compact subset Ω of M, where $\tau(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi$ is the tension field of φ. The Euler-Lagrange equation of this functional gives the biharmonic map equation ([15])

$$(1) \qquad \tau^2(\varphi) := \mathrm{Trace}_g(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla^M})\tau(\varphi) - \mathrm{Trace}_g R^N(\mathrm{d}\varphi, \tau(\varphi))\mathrm{d}\varphi = 0,$$

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which states the fact that the map φ is biharmonic if and only if its bitension field $\tau^2(\varphi)$ vanishes identically. In the above equation we have used R^N to denote the curvature operator of (N,h) defined by

$$R^N(X,Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.$$

Clearly, it follows from (1) that any harmonic map is biharmonic, so we call those non-harmonic biharmonic maps **proper biharmonic maps**.

A submanifold is **biharmonic** if the isometric immersion defining the submanifold is a biharmonic map. It is well known that an isometric immersion is minimal if and only if it is harmonic. So a minimal submanifold is trivially biharmonic and we call those non-minimal biharmonic submanifolds **proper biharmonic submanifolds**.

Examples of proper biharmonic maps are very difficult to find. Most of the known examples come from the following families.

1. **Biharmonic isometric immersions:** **(i)** The generalized Clifford torus $S^p(\frac{1}{\sqrt{2}}) \times S^q(\frac{1}{\sqrt{2}}) \hookrightarrow S^n$ with $p \neq q, p+q=n$ ([17]); **(ii)** The hypersphere $S^n(\frac{1}{\sqrt{2}}) \hookrightarrow S^{n+1}$ ([5]); **(iii)** Biharmonic cylinder : $S^1(\frac{1}{\sqrt{2}}) \times \mathbb{R} \hookrightarrow S^2 \times \mathbb{R}$ ([26]); **(iv)** The hyperplanes $z = k$ (k is a constant $> -C$) in the conformally flat space $(\mathbb{R}^{m+1}_+, h = (\frac{z+C}{D})^2(\sum_{i=1}^m \mathrm{d}x_i^2 + \mathrm{d}z^2))$ ([26]);
2. **Biharmonic conformal immersions:** **(i)** The inversion in 3-sphere: $\phi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4$ with $\phi(x) = \frac{x}{|x|^2}$ ([1]). This is also the only known biharmonic morphism which is not a harmonic morphisms ([20]); **(ii)** Some proper biharmonic identity maps [20]: Euclidean-to-Hyperbolic space, $id : (H^4 = \mathbb{R}^3 \times \mathbb{R}^+, ds^2) \longrightarrow (H^4, x_4^{-2}ds^2)$ and $id : (B^4 = \{x \in \mathbb{R}^4 : |x| < 1\}, ds^2) \longrightarrow (B^4, 4(1-|x|^2)^{-2}ds^2)$; Euclidean-to-spherical space, $id : (\mathbb{R}^4, ds^2) \longrightarrow (S^4 \setminus \{N\}, 4(1+|x|^2)^{-2}ds^2)$. **(iii)** Conformal immersions from Euclidean space into space forms [25]: $\varphi : (\mathbb{R}^3 \times \mathbb{R}^+, \bar{g} = \delta_{ij}) \longrightarrow (H^5 = \mathbb{R}^4 \times \mathbb{R}^+, h = y_5^{-2}\delta_{\alpha\beta})$ with $\varphi(x_1, \ldots, x_4) = (1, x_1, \ldots, x_4)$ and $\varphi : (\mathbb{R}^4, \bar{g} = \delta_{ij}) \longrightarrow (S^5 \setminus \{N\} \equiv \mathbb{R}^5, h = \frac{4\delta_{\alpha\beta}}{(1+|y|^2)^2})$ with $\varphi(u_1, \ldots, u_4) = (u_1, \ldots, u_4, 0)$, where $(u_1, \ldots, u_5)$ are conformal coordinates on $S^5 \setminus \{N\} \equiv \mathbb{R}^5$; **(iv)** The conformal biharmonic surfaces in $\mathbb{R}^3$ [25]: For $\lambda^2 = \big(C_2 e^{\pm z/R} - C_1 C_2^{-1} R^2 e^{\mp z/R}\big)/2$ with constants C_1, C_2, the maps $\phi : (D, g = \lambda^{-2}(R^2 d\theta^2 + dz^2)) \longrightarrow (\mathbb{R}^3, d\sigma^2 = d\rho^2 + \rho^2\, d\theta^2 + dz^2)$ with $\phi(\theta, z) = (R, \theta, z)$ is a family of proper biharmonic conformal immersions of a cylinder of radius R into Euclidean space $\mathbb{R}^3$, where $D = \{(\theta, z) \in (0, 2\pi) \times \mathbb{R} : z \neq \pm\frac{R}{2}\ln(C_1 R^2/C_2^2)\}$.

3. **Biharmonic horizontally weakly conformal maps: (i)** Hopf construction map $\phi : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R} \times \mathbb{R}^2$, $\phi(z,w) = (|z|^2 - |w|^2, 2z\bar{w})$ with the standard Euclidean metric on the domain and certain conformally flat metric on the target space ([27]); **(ii)** The map $\phi : \mathbb{R}^3 \longrightarrow (\mathbb{R}^2, \bar{g})$ with $\phi(x_1, x_2, x_3) = (\sqrt{x_1^2 + x_2^2}\,, x_3)$ and a certain conformally flat metric $\bar{g}$ ([27]); **(iii)** Biharmonic Riemannian submersions [20]:
$\phi : (\mathbb{R}^2 \times \mathbb{R}, dx^2 + dy^2 + \beta^2(x)dz^2) \to (\mathbb{R}^2, dx^2 + dy^2)$, $\phi(x,y,z) = (x,y)$ with $c_1, c_2 \in \mathbb{R}^*$, $\beta = c_2\, e^{\int f(x)\, dx}$, and $f(x) = \frac{-c_1(1+e^{c_1 x})}{1-e^{c_1 x}}$, is a family of biharmonic Riemannian submersions;
4. **Biharmonic maps which are neither conformal immersions nor horizontally weakly conformal maps: (i)** The composition of the Hopf map and the inclusion: $S^3(\frac{1}{\sqrt{2}}) \longrightarrow S^2(\frac{1}{\sqrt{2}}) \hookrightarrow S^3$ ([18]), and more generally, the pull-backs of proper biharmonic maps $S^m(\frac{1}{\sqrt{2}}) \hookrightarrow S^{m+1} (m = 2, 4, 8)$ by the Hopf fibrations $S^{2m-1} \longrightarrow S^m(\frac{1}{\sqrt{2}})$ ([24]); **(ii)** The composition of a harmonic map ψ from a compact manifold with constant energy density and the inclusion map of a biharmonic hypersurface [18]: $i \circ \psi : (M, g) \longrightarrow S^n(\frac{1}{\sqrt{2}}) \hookrightarrow S^{n+1}$; **(iii)** The axially symmetric biharmonic maps [3]: $\phi : (0, \infty)_{t^2} \times S^{m-1} \longrightarrow \mathbb{R} \times_{f^2} S^{n-1}$, $\phi(t,x) = (\rho(t), \varphi_0(x))$, where φ_0 is a constant map; $\phi : \mathbb{R}^m \setminus \{0\} \longrightarrow \mathbb{R}^m \setminus \{0\}$ with $\phi(x) = x/|x|^{m-2}$, $m \neq 4$.

In this paper, we give several construction methods and use them to produce many examples of proper biharmonic maps including biharmonic tori of any dimension in Euclidean spheres (Theorem 2.2, Corollaries 2.3, 2.4 and 2.6), biharmonic maps between spheres (Theorem 2.9) and into spheres (Theorem 2.10) via orthogonal multiplications and eigenmaps. We also study biharmonic graphs of maps, derive the equation for a function whose graph is a biharmonic hypersurface in a Euclidean space, and give an equivalent formulation of Chen's conjecture on biharmonic hypersurfaces by using the biharmonic graph equation (Theorem 4.1) which reveals a similarity to the well-known Bernstein's conjecture on the existence of entire minimal graph. We hope this will pave a way and attract more work and especially more analytic study to Chen's conjecture on biharmonic hypersurfaces.

2. Constructions of proper Biharmonic maps

Let $(M_1 \times M_2, g_1 \times g_2)$ be the Riemannian product of manifolds (M_1^m, g_1) and (M_2^n, g_2). For any map $F : (M_1 \times M_2, g_1 \times g_2) \longrightarrow (N, h)$ with $F = F(x_1, x_2)$ we have two families of maps $F_1, : (M_1^m, g_1) \longrightarrow (N, h)$ with $F_1(x_1) = F(x_1, x_2)$

for fixed $x_2 \in M_2$ and $F_2,: (M_2^n, g_2) \longrightarrow (N, h)$ with $F_2(x_2) = F(x_1, x_2)$ for fixed $x_1 \in M$. Then, we know (cf. [11], Section (4.15)) that F is harmonic if it is harmonic with respect to each variable separately, i.e., both F_1 and F_2 are harmonic. This can be generalized to the case of biharmonic maps as

Proposition 2.1. (Biharmonic maps from product spaces) *A map $F : (M_1 \times M_2, g_1 \times g_2) \longrightarrow (N, h)$ with $F = F(x_1, x_2)$ is biharmonic if both F_1 and F_2 are biharmonic, where, $F_1,: (M_1^m, g_1) \longrightarrow (N, h)$ with $F_1(x_1) = F(x_1, x_2)$ for fixed $x_2 \in M_2$ and $F_2,: (M_2^n, g_2) \longrightarrow (N, h)$ with $F_2(x_2) = F(x_1, x_2)$ for fixed $x_1 \in M$. Furthermore, if one of F_1 and F_2 is proper biharmonic, then F is a proper biharmonic map.*

Proof. Choose a local orthonormal frame $\{e_i\}_{i=1,\ldots,m}$ on M_1 and a local orthonormal frame $\{c_a\}_{a=1,\ldots,n}$ on M_2 so that $\{e_i, c_a\}_{i=1,\ldots,m,\ a=1,\ldots,n}$ form a local orthonormal frame on $M_1 \times M_2$. Let $\pi_k : M_1 \times M_2 \longrightarrow M_k, \pi_k(x_1, x_2) = x_k$, be the projection onto the kth factor $(k = 1, 2)$. Then, it is easily seen that

$$\mathrm{d}F = \mathrm{d}(F_1 \circ \pi_1) + \mathrm{d}(F_2 \circ \pi_2), \tag{2}$$

The tension field of F is given (see e.g., (4.15) in [11]) by

$$\begin{aligned}\tau(F) = \quad & \sum_{i=1}^{m}\{\nabla^{F_1\circ\pi_1}_{e_i}\mathrm{d}(F_1 \circ \pi_1)e_i - \mathrm{d}(F_1 \circ \pi_1)\nabla^{M_1\times M_2}{}_{e_i}e_i\} \\ & + \sum_{a=1}^{n}\{\nabla^{F_2\circ\pi_2}_{c_a}\mathrm{d}(F_2 \circ \pi_2)c_a - \mathrm{d}(F_2 \circ \pi_2)\nabla^{M_1\times M_2}{}_{c_a}c_a\} \\ = \quad & \tau(F_1 \circ \pi_1) + \tau(F_2 \circ \pi_2). \qquad (3)\end{aligned}$$

Using the fact that the Jacobi operator is linear we have

$$\begin{aligned}\tau^2(F) = \quad & -J^F(\tau(F)) = -J^F\left[\tau(F_1 \circ \pi_1) + \tau(F_2 \circ \pi_2)\right] \\ & = -J^F\big(\tau(F_1 \circ \pi_1)\big) - J^F\big(\tau(F_2 \circ \pi_2)\big) \\ & = -J^{F_1\circ\pi_1}\big(\tau(F_1 \circ \pi_1)\big) - J^{F_2\circ\pi_2}\big(\tau(F_2 \circ \pi_2)\big) \\ & = \tau^2(F_1 \circ \pi_1) + \tau^2(F_2 \circ \pi_2) \\ & = \tau^2(F_1) \circ \pi_1 + \tau^2(F_2) \circ \pi_2,\end{aligned} \tag{4}$$

where the last equality is obtained by using the fact that both π_1 and π_2 are harmonic morphisms with constant dilations $\lambda \equiv 1$ and hence biharmonic morphisms ([23]). The proposition follows from (4) and (3). $\square$

Remark 1. It is very easy to see that the converse of Proposition 2.1 is not true. For example, $F : S^1 \times S^1 \longrightarrow S^3$ defined by $F(x, y) = (\frac{\sqrt{3}}{2}\cos x,\ \frac{\sqrt{3}}{2}\sin x,\ \frac{1}{2}\cos y,\ \frac{1}{2}\sin y)$

is harmonic (and hence biharmonic) but it is not biharmonic with respect to either single variable.

Example 1. The map $F : \mathbb{R} \times (\mathbb{R}^4 \setminus \{0\}) \longrightarrow \mathbb{R}^4$ $F(t,x) = \frac{tx}{|x|^2}$ with rational functions as component functions is a proper biharmonic map. The effect of the map can be interpreted as an inversion of the point $x \in \mathbb{R}^4$ about the unit sphere $S^3 \subset \mathbb{R}^4$ followed by a translation of t unit along the direction of x for $t > 0$ (and opposite direction of x for $t < 0$). To see that this map is proper biharmonic we notice that for each fixed t_0, the map $F(t_0, \cdot) : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4$ is a constant multiple of the inversion about 3-sphere which is proper biharmonic by [1]. On the other hand, for each fixed x_0, the map $F(\cdot, x_0) : \mathbb{R} \longrightarrow \mathbb{R}^4$ is a linear map, a straight line and hence a geodesic, which is clearly harmonic. From these and Proposition 2.1 we conclude that the map is indeed proper biharmonic.

Example 2. The map $\phi : S^n(\frac{1}{\sqrt{2}}) \times \mathbb{R} \longrightarrow S^{n+1} \times \mathbb{R}$ with $\phi(x,t) = (x, \frac{1}{\sqrt{2}}, t)$ is a proper biharmonic map. This follows from Proposition 2.1 and the fact that for each fixed t, the map is the inclusion $S^n(\frac{1}{\sqrt{2}}) \hookrightarrow S^{n+1}$ which is proper biharmonic ([5]); and for each fixed $x \in S^n(\frac{1}{\sqrt{2}})$, the map is a geodesic in $S^{n+1} \times \mathbb{R}$ and hence harmonic.

2.1 Biharmonic maps via orthogonal multiplications. An orthogonal multiplication is a bilinear map $f : \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}^r$ such that $|f(x,y)| = |x||y|$. It is well known (see, e.g., [11], Section (4.16)) that any orthogonal multiplication restricts to a bi-eigenmap $f : S^{p-1} \times S^{q-1} \longrightarrow S^{r-1}$ which is totally geodesic embedding and hence a harmonic map in each variable separately. Furthermore, for the orthogonal multiplications $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, n = 1, 2, 4, 8$ defined by the usual multiplications of algebras of real, complex, quaternionic, and Cayley numbers, the Hopf construction maps $F : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ defined by $F(x,y) = (2f(x,y), |x|^2 - |y|^2)$ restrict to the Hopf fibrations $S^{2n-1} \longrightarrow S^n$ which are harmonic maps. The following theorem shows that any orthogonal multiplication can be used to construct a proper biharmonic map from a torus into a sphere.

Theorem 2.2. *For any orthogonal multiplication $f : \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$, the map $\phi : \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}^{n+1}$ defined by $\phi(x,y) = (\frac{1}{\sqrt{2}} f(x,y), \frac{1}{\sqrt{2}})$ restricts to a proper biharmonic map $S^{p-1} \times S^{q-1} \longrightarrow S^n$.*

Proof. It is easily checked that the map does restrict to a map $\phi : S^{p-1} \times S^{q-1} \longrightarrow S^n$. Note that, with respect to x-variable (i.e., for each fixed $y_0 \in S^{q-1}$), the map $\phi(\cdot, y_0) : S^{p-1} \longrightarrow S^n$ is the composition of $\psi : S^{p-1} \longrightarrow S^{n-1}(\frac{1}{\sqrt{2}})$, $\psi(x) = \frac{1}{\sqrt{2}} f(x, y_0)$, followed by the inclusion map $\mathbf{i} : S^{n-1}(\frac{1}{\sqrt{2}}) \hookrightarrow S^n$. Since f is an

orthogonal multiplication, the map ψ is a totally geodesic homothetic imbedding and hence a harmonic map with constant energy density. It follows from Theorem 1.1 in [18] that the map $\phi(\cdot, y_0) : S^{p-1} \longrightarrow S^{n-1}(\frac{1}{\sqrt{2}}) \hookrightarrow S^n$ is a proper biharmonic map. Similarly, with respect to y-variable the map $\phi(x_0, \cdot) : S^{q-1} \longrightarrow S^n$ is also a proper biharmonic map. Applying Proposition 2.1 we obtain the theorem. □

Corollary 2.3. (Biharmonic tori in S^n) *For any $k, n \geq 2$, there is a proper biharmonic map $T^k = S^1 \times \ldots \times S^1 \longrightarrow S^n$ from flat torus into n-sphere. In particular, the map $\phi(t,s) = (\frac{1}{\sqrt{2}}\cos(t+s), \frac{1}{\sqrt{2}}\sin(t+s), \frac{1}{\sqrt{2}})$ is a proper biharmonic map from 2-torus T^2 into 2-sphere S^2.*

Proof. The proper biharmonic flat torus $\phi : S^1 \times S^1 \longrightarrow S^2$ is obtained by applying Theorem 2.2 with the orthogonal multiplication $f : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by the product of complex numbers, i.e., $f(z,w) = zw$. In fact, let $(z,w) = (e^{it}, e^{is}) \in S^1 \times S^1$. then, By Theorem 2.2, the map can be expressed as $\phi(t,s) = (\frac{1}{\sqrt{2}}\cos(t+s), \frac{1}{\sqrt{2}}\sin(t+s), \frac{1}{\sqrt{2}})$. Note that with respect to each variable the map ϕ is a proper biharmonic curve (up to an affine transformation of arc-length parameter) in S^2 ([5]). By totally geodesically embedding S^2 into S^n and a result in [24] stating that totally geodesically immersing the target manifold of a biharmonic map into another manifold does not change the biharmonicity of the map we obtain the flat torus in S^n. □

Remark 2. Note that the proper biharmonic map $T^2 \longrightarrow S^2$ constructed in Corollary 2.3 is not onto so the degree of the map is 0. It would be interesting to know if there exists proper biharmonic map of degree ± 1 from T^2 to S^2 as it was showed by J. Eells and J.C. Wood [13] that there exists no harmonic map from T^2 to S^2 (whatever the metrics chosen) in the homotopy class of Brower degree ± 1.

Corollary 2.4. (**A flat torus in** S^4) *The map $\phi : \phi : S^1 \times S^1 \longrightarrow S^4$ defined by*

$$\phi(t,s) = \frac{1}{\sqrt{2}}(\cos t \cos s, \quad \cos t \sin s, \quad \sin t \cos s, \quad \sin t \sin s, \quad 1)$$

is a proper biharmonic map.

Proof. This is obtained by applying Theorem 2.2 with the orthogonal multiplication $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ defined (c.f., [28]) by $f(x,y) = (x_1y_1, x_2y_1, x_1y_2, x_2y_2)$. Note that in this case, there is an interesting way to see that the map is a proper biharmonic curve in S^4 with respect to each variable separately. For instance, with respect to t-variable (s is fixed), the map becomes a curve $\phi(t,s) =$

$c_1 \cos t + c_2 \sin t + c_3$, where

$$\begin{aligned} c_1 &= \frac{1}{\sqrt{2}}(\cos s, \sin s, 0, 0, 0), \quad c_2 = \frac{1}{\sqrt{2}}(0, 0, \cos s, \sin s, 0), \\ c_3 &= (0, 0, 0, 0, \frac{1}{\sqrt{2}}) \end{aligned}$$

are three mutually orthogonal vectors in $\mathbb{R}^5$. It follows from Proposition 4.4 in [6] that the t-curve is a proper biharmonic curve in S^4 (up to an affine transformation of arc-length parameter). Similarly, with respect to s-variable, the map ϕ is also a proper biharmonic curve. □

Remark 3. (1) A similar construction applies to the family of the orthogonal multiplications $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ defined in [28] will produce a family of proper flat tori in S^4.
(2) Note that none of the proper biharmonic maps constructed above is an isometric immersion. So they are different from the closed orientable proper biharmonic embedded surfaces of any genus in S^4 described in [6].

It is also known (see, e.g., [11], Section (4.16)) that for any bi-eigenmap $f : S^{p-1} \times S^{q-1} \longrightarrow S^{n-1}$, and harmonic maps $\varphi : M \longrightarrow S^{p-1}$ and $\psi : N \longrightarrow S^{q-1}$, then the map $f \circ (\varphi, \psi) : M \times N \longrightarrow S^{n-1}$ is also harmonic. We have the following generalization.

Proposition 2.5. *Let $f : \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$ be an orthogonal multiplication, and $\varphi : M \longrightarrow S^{p-1}$ and $\psi : N \longrightarrow S^{q-1}$ be biharmonic maps, then the map $f \circ (\varphi, \psi) : M \times N \longrightarrow S^{n-1}$ is biharmonic. If one of φ and ψ is proper biharmonic, then so is the map $f \circ (\varphi, \psi) : M \times N \longrightarrow S^{n-1}$.*

Proof. If $f : \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$ is an orthogonal multiplication, then the bi-eigenmap $f : S^{p-1} \times S^{q-1} \longrightarrow S^{n-1}$ is totally geodesic embedding in each variable separately. Notice that with respect to x-variable, the map $f \circ (\varphi, \psi)(\cdot, y_0) = f(\cdot, \psi(y_0)) \circ \varphi$, a composition of a proper biharmonic map followed by a totally geodesic imbedding. It follows from Proposition 4.4 in [24] that the map $f \circ (\varphi, \psi)$ is a proper biharmonic map with respect to x-variable. Similarly, $f \circ (\varphi, \psi)$ is also proper biharmonic with respect to y-variable. These, together with Proposition 2.1, give the required results. □

Corollary 2.6. (Biharmonic tori in S^3) *For any $k \geq 2$, there is a proper biharmonic map from flat torus into 3-sphere, $T^k = S^1 \times \ldots \times S^1 \longrightarrow S^3$. In particular, the maps $\phi, \xi : S^1 \times S^1 \longrightarrow S^3$,*

$$\phi(t, s) = \frac{1}{2}\big(-\sin t - \cos s, \ \cos t + \sin s, \ \cos (t+s), \ \sin (t+s) - 1 \big), \tag{5}$$

and

$$\xi(t,s) = \big(\cos^2\frac{t+s}{2},\ \sin\frac{t+s}{2}\cos\frac{t+s}{2}, \sin\frac{t+s}{2}\sin\frac{t-s}{2}, -\sin\frac{t+s}{2}\cos\frac{t-s}{2}\big), \tag{6}$$

define two proper biharmonic flat tori in S^3.

Proof. For the last statement, let $f:\mathbb{R}^4\times\mathbb{R}^4\longrightarrow\mathbb{R}^4$ be the orthogonal multiplication defined by the quaternion product

$$\begin{aligned} f((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)) &= xy \\ = \quad & (x_1y_1-x_2y_2-x_3y_3-x_4y_4,\ \ x_1y_2+x_2y_1+x_3y_4-x_4y_3, \\ & x_1y_3-x_2y_4+x_3y_1+x_4y_2,\ \ x_1y_4+x_2y_3-x_3y_2+x_4y_1). \end{aligned} \tag{7}$$

It restricts to a bi-eigenmap $f:S^3\times S^3\longrightarrow S^3$. Let $\varphi,\psi,\sigma:S^1\longrightarrow S^3$ be the non-geodesic biharmonic curves (up to an affine transformation of arclength parameter) defined ([5]) by $\varphi(t)=(\frac{\cos t}{\sqrt{2}},\frac{\sin t}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$, $\psi(s)=(0,\frac{1}{\sqrt{2}},\frac{\cos s}{\sqrt{2}},\frac{\sin s}{\sqrt{2}})$, and $\sigma(s)=(\frac{\cos s}{\sqrt{2}},\frac{\sin s}{\sqrt{2}},-\frac{1}{\sqrt{2}},0)$ respectively. Then, by Proposition 2.5, we obtain a proper biharmonic map
$f\circ(\varphi,\psi):S^1\times S^1\longrightarrow S^3$ defined by

$$\begin{aligned} f\circ\big(\varphi,\psi)(t,s) &= (\frac{\cos t}{\sqrt{2}},\frac{\sin t}{\sqrt{2}},\frac{1}{\sqrt{2}},0)(0,\frac{1}{\sqrt{2}},\frac{\cos s}{\sqrt{2}},\frac{\sin s}{\sqrt{2}}) \\ &= \frac{1}{2}\big(-\sin t-\cos s,\ \ \cos t+\sin s,\ \cos(t+s),\ \ \sin(t+s)-1\big). \end{aligned} \tag{8}$$

A straightforward computation shows that $\xi(t,s)=f\circ(\varphi,\sigma)$ which is also a proper biharmonic map by Proposition 2.5.

Applying a similar construction to k multi-eigenmap $S^3\times\cdots\times S^3\longrightarrow S^3$ we obtain the biharmonic flat tori T^k in S^3 which gives the first statement of the corollary. □

Remark 4. (i) We would like to point out that, by a classification theorem in [5], the only proper biharmonic isometric immersion from a surface into S^3 is the inclusion map of (a part of) $S^2(\frac{1}{\sqrt{2}})\subset S^3$. Our proper biharmonic maps (5) and (6) from flat torus into S^3 is clearly not isometric immersions.
(ii) We also note that there are many harmonic maps from flat torus into S^3. For example, one can check that the following family (in parameter s) of maps: φ_s : $S^1\times S^1\longrightarrow S^3$ defined by $\varphi_s(x,y)=(\cos s\,\cos x,\ \ \cos s\,\sin x,\ \ \sin s\,\cos y,\ \ \sin s\,\sin y)$ are harmonic maps for all $s\in[0,\pi/2]$ (see [2] for details).

Example 3. **Proper biharmonic maps from $S^2\times S^2$, $S^2\times S^1$ and $S^2\times S^3$ into S^3.**

Let $f: S^3\times S^3 \longrightarrow S^3$ be the bi-eigenmap defined by the product of quaternion. Let $\varphi: S^2\longrightarrow S^3$ be the biharmonic map defined ([5]) by $\varphi(x_1,x_2,x_3) = (\frac{x_1}{\sqrt{2}}, \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $\psi: S^1\longrightarrow S^3$ be the geodesic defined by $\psi(y_1,y_2)=(y_1,y_2,0,0)$, and $i: S^3\longrightarrow S^3$ be the identity map defined by $i(y_1,y_2,y_3,y_4)=(y_1,y_2,y_3,y_4)$. Then, it follows from Proposition 2.5 that each of the following three maps into 3-sphere is a proper biharmonic map:

(a) $f\circ(\varphi,\varphi): S^2\times S^2\longrightarrow S^3$ with

$$f\circ(\varphi,\varphi)(x_1,x_2,x_3,y_1,y_2,y_3) = (\frac{x_1}{\sqrt{2}}, \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{2}}, \frac{1}{\sqrt{2}})(\frac{y_1}{\sqrt{2}}, \frac{y_2}{\sqrt{2}}, \frac{y_3}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \tag{9}$$

$$= \frac{1}{2}(x_1y_1 - x_2y_2 - x_3y_3 - 1,\ \ x_1y_2 + x_2y_1 + x_3 - y_3,\ \ x_1y_3 + x_3y_1 - x_2 + y_2,$$
$$x_2y_3 - x_3y_2 + x_1 + y_1),$$

(b) $f\circ(\varphi,\psi): S^2\times S^1\longrightarrow S^3$ with

$$f\circ(\varphi,\psi)(x_1,x_2,x_3,y_1,y_2) = (\frac{x_1}{\sqrt{2}}, \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{2}}, \frac{1}{\sqrt{2}})(y_1,y_2,0,0) \tag{10}$$

$$= \frac{1}{\sqrt{2}}(x_1y_1 - x_2y_2,\ \ x_1y_2 + x_2y_1,\ \ x_3y_1 + y_2,\ \ -x_3y_2 + y_1),$$

(c) $f\circ(\varphi,i): S^2\times S^3\longrightarrow S^3$ with

$$f\circ(\varphi,i)(x_1,x_2,x_3,y_1,y_2,y_3,y_4) = (\frac{x_1}{\sqrt{2}}, \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{2}}, \frac{1}{\sqrt{2}})(y_1,y_2,y_3,y_4) \tag{11}$$

$$= \frac{1}{\sqrt{2}}(x_1y_1 - x_2y_2 - x_3y_3 - y_4,\ \ x_1y_2 + x_2y_1 + x_3y_4 - y_3,$$
$$x_1y_3 + x_3y_1 - x_2y_4 + y_2,\ \ \ x_2y_3 - x_3y_2 + x_1y_4 + y_1).$$

Corollary 2.7. *Let G be Lie group with bi-invariant Riemannian metric, and $f: G\times G\longrightarrow G$ be the product of the Lie group. Let $\varphi: M\longrightarrow G$ and $\psi: N\longrightarrow G$ be biharmonic maps. Then, the map $f\circ(\varphi,\psi): M\times N\longrightarrow G$ is a biharmonic map. If one of φ and ψ is proper biharmonic, then so is the map $f\circ(\varphi,\psi): M\times N\longrightarrow G$.*

Proof. It is easy to see that the product map f of the Lie group is an isometry with respect to each variable separately. Noting that each of these isometries is an isometry between the same dimensional manifolds we conclude that f is totally geodesic with respect to each variable separately. An argument similar to

the one used in the proof of Proposition 2.5 applies to give the statements of the corollary. □

The following proposition shows that orthogonal multiplications can also be used to construct proper biharmonic maps into Euclidean space.

Proposition 2.8. *Let $f : \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$ be an orthogonal multiplication, and $\varphi : M \longrightarrow \mathbb{R}^p$ and $\psi : N \longrightarrow \mathbb{R}^q$ be biharmonic maps, then the map $f \circ (\varphi, \psi) : M \times N \longrightarrow \mathbb{R}^n$ is also a biharmonic map. Furthermore, if one of φ and ψ is proper biharmonic, then the map $f \circ (\varphi, \psi) : M \times N \longrightarrow \mathbb{R}^n$ is proper biharmonic.*

Proof. This follows from an argument similar to the one used in the proof of Proposition 2.5 and the fact that with respect to each variable, the orthogonal multiplication $f : \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$ is a homothetic totally geodesic imbedding. □

Example 4. Let $f : \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$, $f(x, y) = xy$ be the orthogonal multiplication defined by the product of quaternions. Let $\varphi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4$ be the inversion about 3-sphere. Then, by Proposition 2.8, we have a proper biharmonic map defined by rational functions: $f \circ (\varphi, \varphi) : (\mathbb{R}^4 \setminus \{0\}) \times (\mathbb{R}^4 \setminus \{0\}) \longrightarrow \mathbb{R}^4$, $f \circ (\varphi, \varphi)(x, y) = \frac{xy}{|x|^2|y|^2}$, where xy denotes the quaternion product of x and y.

Example 5. Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^4$, be the orthogonal multiplication defined by the restriction of the product of quaternions, i.e.,

$$\begin{aligned} f((x_1, x_2, x_3), (y_1, y_2, y_3))) &= (x_1, x_2, x_3, 0), (y_1, y_2, y_3, 0)) \\ = \quad & (x_1y_1 - x_2y_2 - x_3y_3,\ \ x_1y_2 + x_2y_1,\ \ x_1y_3 + x_3y_1,\ \ x_2y_3 - x_3y_2) \end{aligned} \tag{12}$$

Let $\varphi : (\mathbb{R}^2, g = e^{x_2}(dx_1^2 + dx_2^2)) \longrightarrow \mathbb{R}^3$, $\varphi(x_1, x_2) = (\cos x_1, \sin x_1, x_2)$ be the proper biharmonic conformal immersion of $\mathbb{R}^2$ into Euclidean space $\mathbb{R}^3$ ([25]). Then, by Proposition 2.8, we have a proper biharmonic map
$f \circ (\varphi, \varphi) : (\mathbb{R}^4, e^{x_2}(dx_1^2 + dx_2^2) + e^{y_2}(dy_1^2 + dy_2^2)) \longrightarrow \mathbb{R}^4$ defined by

$$\begin{aligned} & f \circ (\varphi, \varphi)(x, y) = \\ & (\cos(x_1 + y_1) - x_2y_2, \sin(x_1 + y_1), y_2 \cos x_1 + x_2 \cos y_1, y_2 \sin x_1 - x_2 \sin y_1). \end{aligned}$$

2.2 More on biharmonic maps between spheres and into spheres

There are many harmonic maps between spheres which include the standard minimal isometric immersions (embeddings) of spheres into spheres. To the author's knowledge, the only known examples of proper biharmonic maps between Euclidean spheres seem to be the following:

(1) the inclusion map ([5]) $\mathbf{i} : S^m(\frac{1}{\sqrt{2}}) \hookrightarrow S^{m+1}$, $\mathbf{i}(x) = (x, \frac{1}{\sqrt{2}})$ for $x \in \mathbb{R}^{m+1}$ with $|x| = \frac{1}{\sqrt{2}}$, or more generally, homothetic immersion $\phi : S^m \hookrightarrow S^{m+1}$, $\mathbf{i}(x) = (\frac{x}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ for $x \in \mathbb{R}^{m+1}$ with $|x| = 1$ (see [24] for details);

(2) The composition of minimal isometric immersion $S^k(\frac{1}{\sqrt{2}}) \hookrightarrow S^m(\frac{1}{\sqrt{2}})$ followed by the inclusion $S^m(\frac{1}{\sqrt{2}}) \hookrightarrow S^{m+1}$ ([6], [19]), the composition of Veronese map: $S^2(\sqrt{3}) \hookrightarrow S^4(\frac{1}{\sqrt{2}}) \hookrightarrow S^5$ ([6]);
(3) The composition of the Hopf map and the inclusion: $S^3(\frac{1}{\sqrt{2}}) \longrightarrow S^2(\frac{1}{\sqrt{2}}) \hookrightarrow S^3$ ([18]), and more generally, the pull-backs of proper biharmonic maps $S^m(\frac{1}{\sqrt{2}}) \hookrightarrow S^{m+1}(m = 2, 4, 8)$ by the Hopf fibrations $S^{2m-1} \longrightarrow S^m(\frac{1}{\sqrt{2}})$ ([24]).

The following construction can be used to produce many proper biharmonic maps between Euclidean spheres.

Theorem 2.9. *For any eigenmap $f : S^{m-1} \longrightarrow S^{n-1}$ there is an associated proper biharmonic map $F : S^{m-1} \longrightarrow S^n$ defined by $F(x) = (\frac{1}{\sqrt{2}}f(x), \frac{1}{\sqrt{2}})$.*

Proof. It is well known that any eigenmap $f : S^{m-1} \longrightarrow S^{n-1}$ comes from the restriction of a homogenous polynomial map (denoted by the same f) $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$. Notice that the map F is the composition of the map $\psi : S^{m-1} \longrightarrow S^{n-1}(\frac{1}{\sqrt{2}})$, $\psi(x) = \frac{1}{\sqrt{2}}f(x)$, followed by the inclusion $\mathbf{i} : S^{n-1}(\frac{1}{\sqrt{2}}) \hookrightarrow S^n$, $\mathbf{i}(x) = (x, \frac{1}{\sqrt{2}})$ for $x \in \mathbb{R}^n$ with $|x| = \frac{1}{\sqrt{2}}$. Let $\rho : S^{n-1} \longrightarrow S^{n-1}(\frac{1}{\sqrt{2}})$ denote the homothetic map defined by $\rho(y) = \frac{1}{\sqrt{2}}y$ for $y \in \mathbb{R}^n$ with $|y| = 1$. Then, $\psi = \rho \circ f$. Since the homothety ρ is a totally geodesic map, a well-known fact about the composition of harmonic maps implies that $\psi = \rho \circ f$ is harmonic. On the other hand, it is not difficult to check that the energy density $e(\psi) = \frac{1}{2}e(f)$ is a constant since $e(f)$ is constant by the definition of an eigenmap. By Theorem 1.1 in [18] we conclude that the map F is a proper biharmonic map. □

Remark 5. As we can see from Theorem 2.9 and the introduction in this section that all known examples of proper biharmonic maps between spheres come from the construction given in Theorem 2.9. It would be interesting to know whether there is other type of proper biharmonic maps between spheres.

Theorem 2.10. *Let $\varphi : M \longrightarrow S^p$ and $\psi : N \longrightarrow S^q$ be harmonic maps with constant energy density from compact manifolds, and $\mathbf{j} : S^p(\frac{1}{\sqrt{2}}) \times S^q(\frac{1}{\sqrt{2}}) \hookrightarrow S^{p+q+1}, p \neq q$, be the standard inclusion. Then, $\phi = \mathbf{j} \circ (\frac{\varphi}{\sqrt{2}}, \frac{\psi}{\sqrt{2}}) : M \times N \longrightarrow S^{p+q+1}$ is a proper biharmonic map.*

Proof. For fixed $y_0 \in N$, the map $\phi(x, y_0) = (\frac{\varphi(x)}{\sqrt{2}}, \frac{\psi(y_0)}{\sqrt{2}})$. It is not difficulty to see that there is an isometry of $\sigma : S^{p+q+1} \longrightarrow S^{p+q+1}$ such that the composition $\sigma\circ\phi(x, y_0) = (\varphi(x), \frac{1}{\sqrt{2}}, 0) \in \mathbb{R}^{p+1}\times\mathbb{R}\times\mathbb{R}^q$ which can be viewed as the composition $\rho \circ \mathbf{i} \circ \varphi : (M, g) \longrightarrow S^p(\frac{1}{\sqrt{2}}) \hookrightarrow S^{p+1} \hookrightarrow S^{p+q+1}$ with $\mathbf{i} : S^p(\frac{1}{\sqrt{2}}) \hookrightarrow S^{p+1}$ being the standard inclusion, $\rho : S^{p+1} \hookrightarrow S^{p+q+1}$ being a totally geodesic imbedding. It

follows that $\rho\circ\mathbf{i}\circ\varphi$ is a proper biharmonic map, i.e., ϕ is proper biharmonic with respect to x-variable. A similar argument shows that ϕ is also proper biharmonic with respect to y-variable. Therefore, by Proposition 2.1, ϕ is a proper biharmonic map. $\square$

Corollary 2.11. (1) *Let $\varphi : S^m \longrightarrow S^p$ and $\psi : S^n \longrightarrow S^q$ be any eigenmaps, then the map $\phi : S^m \times S^n \longrightarrow S^{p+q+1}$ with $\phi(x, y) = \frac{1}{\sqrt{2}}(\varphi(x), \psi(y))$ is a proper biharmonic map for $p \neq q$;*
(2) *Let $\varphi : S^{m_1}\times S^{m_2} \longrightarrow S^p$ and $\psi : S^{n_1}\times S^{n_2} \longrightarrow S^q$ be any bi-eigenmaps defined by orthogonal multiplications, then the map $\phi : S^{m_1} \times S^{m_2} \times S^{n_1} \times S^{n_2} \longrightarrow S^{p+q+1}$ with $\phi(x_1, x_2, y_1, y_2) = \frac{1}{\sqrt{2}}(\varphi(x_1, x_2),\ \ \psi(y_1, y_2))$ is a proper biharmonic map for $p \neq q$;*

Remark 6. When $\varphi : S^k(\frac{1}{\sqrt{2}}) \longrightarrow S^p,\ \varphi(x) = \sqrt{2}\,x$ and $\psi : S^k(\frac{1}{\sqrt{2}}) \longrightarrow S^q,\ \psi(y) = \sqrt{2}y$ be the homothetic minimal embedding of spheres into spheres, our Theorem 2.10 recover part of results in Proposition 3.10 and Theorem 3.11 in [6]

Example 6. Let $\varphi : S^1 \longrightarrow S^1$, $\varphi(z) = z, z \in \mathbb{C}$ be the identity map and $\psi : S^3 \longrightarrow S^2$ be the Hopf fibration defined by $\psi(x, y) = (2xy, |x|^2 - |y|^2), x, y \in \mathbb{C}$. Then, by Theorem 2.10, we have a proper biharmonic map $\phi : S^1 \times S^3 \longrightarrow S^4$ with $\phi(z, x, y) = \frac{1}{\sqrt{2}}(z, 2xy, |x|^2 - |y|^2)$.

Example 7. Let $\varphi : S^1 \longrightarrow S^1$, $\varphi(z) = z, z \in \mathbb{C}$ be the identity map and $\psi : S^2 \longrightarrow S^2$ be a family of isometries (with parameter t) defined by $\psi(y_1, y_2, y_3) = (y_1 \cos t + y_2 \sin t,\ \ -y_1 \sin t + y_2 \cos t,\ \ y_3), y \in S^2$. Then, by Theorem 2.10, we have a family of proper biharmonic map $\phi : S^1 \times S^2 \longrightarrow S^4$ with $\phi_t(z, y) = \frac{1}{\sqrt{2}}(z, y_1 \cos t + y_2 \sin t,\ \ -y_1 \sin t + y_2 \cos t,\ \ y_3)$. In particular, when $t = 0$, the proper biahrmonic map becomes $\phi_0(z, y) = \frac{1}{\sqrt{2}}(z,\ \ y)$, the standard homothetic embedding which is the composition $S^1 \times S^2 \hookrightarrow S^1(\frac{1}{\sqrt{2}}) \times S^2(\frac{1}{\sqrt{2}}) \hookrightarrow S^4$.

For classification results on proper biharmonic isometric immersions into spheres, i.e., proper biharmonic submanifolds in spheres see a recent survey in [4].

2.3 Biharmonic maps by complete lifts. Let $\varphi : \mathbb{R}^m \supseteq U \longrightarrow \mathbb{R}^n$ be a map into Euclidean space. The complete lift of φ is defined in [22] to be a map $\phi : U \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ with $\phi(x, y) = (\mathrm{d}_x\varphi)(y)$. It was proved in [22] that the complete lift of a harmonic map is a harmonic map, the complete lift of a quadratic harmonic morphism is again a quadratic harmonic morphism. We will show that the complete lift method can also be used to produce new biharmonic maps from given ones.

Proposition 2.12. *The complete lift of a proper biharmonic map is a proper biharmonic map.*

Proof. Let $\varphi(x) = (\varphi^1(x), \varphi^2(x), \ldots, \varphi^n(x))$, then, by definition, $\phi(x, y) = (\sum_{i=1}^m \frac{\partial\varphi^1}{\partial x_i} y_i, \sum_{i=1}^m \frac{\partial\varphi^2}{\partial x_i} y_i, \ldots, \sum_{i=1}^m \frac{\partial\varphi^n}{\partial x_i} y_i)$. It is easy to check that a map into Euclidean space is biharmonic if and only if each of its component function is biharmonic. Let $\Delta_{(x,y)}$ denote the Laplacian on $U \times \mathbb{R}^m$, clearly, $\Delta_{(x,y)} = \Delta_x + \Delta_y$. Therefore,

$$\begin{aligned} \Delta^2_{(x,y)}\phi^a(x,y) &= \Delta_{(x,y)}\big(\Delta_{(x,y)}(\sum_{i=1}^m \frac{\partial\varphi^a}{\partial x_i} y_i)\big) \\ &= \sum_{i=1}^m \Delta_x^2(\frac{\partial\varphi^a}{\partial x_i}) y_i = \sum_{i=1}^m \frac{\partial}{\partial x_i}\big(\Delta_x^2\varphi^a\big) y_i, \end{aligned} \tag{13}$$

from which we obtain the proposition. □

Example 8. Let $\varphi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4$ be the inversion about 3-sphere in $\mathbb{R}^4$ defined by $\varphi(x) = x/|x|^2$ which is known ([1]) to be a proper biharmonic map. Its complete lift $\phi : \mathbb{R}^4 \setminus \{0\} \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is defined by

$$\begin{aligned} \phi(x,y) = |x|^{-4}\big((|x|^2 - 2x_1^2)y_1 - 2x_1 \sum_{i\neq 1}^4 x_i y_i,\ \ (|x|^2 - 2x_2^2)y_2 - 2x_2 \sum_{i\neq 2}^4 x_i y_i, \\ (|x|^2 - 2x_3^2)y_3 - 2x_3 \sum_{i\neq 3}^4 x_i y_i,\ \ (|x|^2 - x_4^2)y_4 - 2x_4 \sum_{i\neq 4}^4 x_i y_i\big), \end{aligned} \tag{14}$$

which is a proper biharmonic map by Proposition 2.12. Note that this is another example of proper biharmonic map defined by rational functions.

2.4 Biharmonic maps by direct sum construction. Let $\varphi : M \longrightarrow \mathbb{R}^n$ and $\psi : N \longrightarrow \mathbb{R}^n$ be two maps. Then the **direct sum** of φ and ψ is defined (see [21] and also [2]) to be the map

$$\varphi \oplus \psi : M \times N \longrightarrow \mathbb{R}^n$$

given by

$$(\varphi \oplus \psi)(p, q) = \varphi(p) + \psi(q)$$

where $M \times N$ is the product of M and N, endowed with the Riemannian product metric $G = g \times h$. An immediate consequence of Proposition 2.1 is the following

Corollary 2.13. *The direct sum of biharmonic maps is again a biharmonic map; the direct sum of a harmonic map and a proper biharmonic map is a proper biharmonic map.*

Example 9. The map $\phi : (\mathbb{R}^3, g = e^{x_2}(dx_1^2 + dx_2^2) + dx_3^2) \longrightarrow \mathbb{R}^3$ defined by $\phi(x_1, x_2, x_3) = (\cos\, x_1 + 3x_3,\ \sin\, x_1 + 2x_3,\ x_2 - x_3)$ is a proper biharmonic map. This follows from Corollary 2.13 and the fact that the map ϕ is the direct sum of the the geodesic (map) $\psi : \mathbb{R}^1 \longrightarrow \mathbb{R}^3$ defined by $\psi(x_3) = (3x_3, 2x_3, -x_3)$ and the map $\varphi : (\mathbb{R}^2, g = e^{x_2}(dx_1^2 + dx_2^2)) \longrightarrow \mathbb{R}^3$, $\varphi(x_1, x_2) = (\cos\, x_1, \sin\, x_1, x_2)$, which is a proper biharmonic conformal immersion of $\mathbb{R}^2$ into Euclidean space $\mathbb{R}^3$ ([25]).

3. Biharmonic maps into a product manifold

The graph of a map $\psi : (M, g) \longrightarrow (N, h)$ is defined to be the map $\phi : M \longrightarrow (M \times N, g \times h)$ with $\phi(x) = (x, \psi(x))$ which is easily checked to be an embedding. So with respect to the pull-back metric $\phi^*(g + h) = g + \psi^* h$ the graph ϕ is an isometric embedding whilst with respect to the original metric g on M it need not be so. The harmonicity of the graph of a map ψ with respect to the pull-back metric $\phi^*(g + h) = g + \psi^* h$ and the original metric g on M were studied in [12] and [10].

Proposition 3.1. ([12], *2 (E)*) *Let $\psi : (M, g) \longrightarrow (N, h)$ be a map. Then, the graph $\phi : (M, g) \longrightarrow (M \times N, g \times h)$ with $\phi(x) = (x, \psi(x))$ is a harmonic map if and only if the map $\psi : (M, g) \longrightarrow (N, h)$ is a harmonic map.*

Proposition 3.2. ([10]) *Let $\psi : (M, g) \longrightarrow (N, h)$ be a map. Then, the graph $\phi : (M, \phi^*(g \times h)) \longrightarrow (M \times N, g \times h)$ with $\phi(x) = (x, \psi(x))$ is a harmonic map if and only if both $\pi_1 \circ \phi$ and $\pi_2 \circ \phi = \psi$ are harmonic, where $\pi_1 : (M \times N, g \times h) \longrightarrow (M, g)$ and $\pi_2 : (M \times N, g \times h) \longrightarrow (N, h)$ are the projections onto the first and the second factor respectively.*

We will show that these results generalize to the case of biharmonic maps. Actually, we will prove that the generalizations follow from the following theorem.

Theorem 3.3. (Biharmonic maps into product spaces) *Let $\varphi : (M, g) \longrightarrow (N_1, h_1)$ and $\psi : (M, g) \longrightarrow (N_2, h_2)$ be two maps. Then, the map $\phi : (M, g) \longrightarrow (N_1 \times N_2, h_1 \times h_2)$ with $\phi(x) = (\varphi(x), \psi(x))$ is biharmonic if and only if both map φ and ψ are biharmonic. Furthermore, if one of φ and ψ is harmonic and the other is a proper biharmonic map, then ϕ is a proper biharmonic map.*

Proof. It is easily seen that

$$\mathrm{d}\phi(X) = \mathrm{d}\varphi(X) + \mathrm{d}\psi(X),\ \ \forall\, X \in TM. \tag{15}$$

It follows that

$$\nabla^{\phi}_{X}\mathrm{d}\phi(Y) = \nabla^{\phi}_{X}\mathrm{d}\varphi(Y) + \nabla^{\phi}_{X}\mathrm{d}\psi(Y),\ \ \forall\, X,\ Y \in TM. \tag{16}$$

Let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame on (M, g) and let $Y = Y^i e_i$, then $\mathrm{d}\varphi(Y) = Y^i \varphi_i^a (E_a \circ \varphi)$ for some function φ_i^a defined locally on M. A straightforward computation yields

$$
\begin{aligned}
\nabla^{\phi}_X \mathrm{d}\varphi(Y) &= \nabla^{\phi}_X (Y^i \varphi_i^a)(E_a \circ \varphi) \\
&= [X(Y^i \varphi_i^a)](E_a \circ \varphi) + (Y^i \varphi_i^a) \nabla^{N_1}_{\mathrm{d}\varphi(X)} (E_a \circ \varphi) \\
&= \nabla^{\varphi}_X (Y^i \varphi_i^a)(E_a \circ \varphi) = \nabla^{\varphi}_X \mathrm{d}\varphi(Y).
\end{aligned} \tag{17}
$$

Similarly, we have

$$
\nabla^{\phi}_X \mathrm{d}\psi(Y) = \nabla^{\psi}_X \mathrm{d}\psi(Y). \tag{18}
$$

By using (15), (16), (17), and (18) we have

$$
\begin{aligned}
\tau(\phi) &= \sum_{i=1}^{m} \{\nabla^{\phi}_{e_i} \mathrm{d}\phi(e_i) - \mathrm{d}\phi(\nabla^M_{e_i} e_i)\} \\
&= \sum_{i=1}^{m} \{\nabla^{\varphi}_{e_i} \mathrm{d}\varphi(e_i) - \mathrm{d}\varphi(\nabla^M_{e_i} e_i) + \nabla^{\psi}_{e_i} \mathrm{d}\psi(e_i) - \mathrm{d}\psi(\nabla^M_{e_i} e_i)\} \\
&= \tau(\varphi) + \tau(\psi).
\end{aligned} \tag{19}
$$

To compute the bitension field of the map ϕ, we notice that $\tau(\varphi)$ is tangent to N_1 whilst $\tau(\psi)$ is tangent to N_2. We use the property of the curvature of the product manifold to have

$$
\begin{aligned}
& \mathrm{R}^{N_1 \times N_2}(\mathrm{d}\phi(e_i), \tau(\phi))\mathrm{d}\phi(e_i) \\
=\ & \mathrm{R}^{N_1}(\mathrm{d}\varphi(e_i), \tau(\varphi))\mathrm{d}\varphi(e_i) + \mathrm{R}^{N_2}(\mathrm{d}(\mathrm{d}\psi(e_i), \tau(\psi))\mathrm{d}\psi(e_i).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tau^2(\phi) &= \sum_{i=1}^{m} \{\nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} \tau(\phi) - \nabla^{\phi}_{\nabla^M_{e_i} e_i} \tau(\phi) - \mathrm{R}^{N_1 \times N_2}(\mathrm{d}\phi(e_i), \tau(\varphi))\mathrm{d}\phi(e_i)\} \\
&= \sum_{i=1}^{m} \{\nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} \tau(\varphi) - \nabla^{\varphi}_{\nabla^M_{e_i} e_i} \tau(\varphi) - \mathrm{R}^{N_1}(\mathrm{d}\varphi(e_i), \tau(\varphi))\mathrm{d}\varphi(e_i)\} \\
&\quad + \sum_{i=1}^{m} \{\nabla^{\psi}_{e_i} \nabla^{\psi}_{e_i} \tau(\psi) - \nabla^{\psi}_{\nabla^M_{e_i} e_i} \tau(\psi) - \mathrm{R}^{N_2}(\mathrm{d}\psi(e_i), \tau(\psi))\mathrm{d}\psi(e_i)\} \\
&= \tau^2(\varphi) + \tau^2(\psi),
\end{aligned} \tag{20}
$$

from which, together with (19) the theorem follows. □

The following corollary generalizes Proposition 3.1 in [12] and can be used to produce proper biharmonic maps from given ones.

Corollary 3.4. *Let $\psi : (M, g) \longrightarrow (N, h)$ be a map. Then, the graph $\phi : (M, g) \longrightarrow (M \times N, g \times h)$ with $\phi(x) = (x, \psi(x))$ is a biharmonic map if and only if the map $\psi : (M, g) \longrightarrow (N, h)$ is a biharmonic map. Furthermore, if ψ is proper biharmonic, then so is the graph.*

Proof. The corollary follows from Theorem 3.3 with $\varphi : (M, g) \longrightarrow (M, g)$ being identity map which is harmonic. □

Example 10. The map $\phi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4 \times \mathbb{R}^4$ given by $\phi(x) = (x, x/|x|^2)$ is a proper biharmonic map. This follows from Theorem 3.3 and the fact that ϕ is the graph of the inversion $\psi : \mathbb{R}^4 \setminus \{0\} \longrightarrow \mathbb{R}^4$ defined by $\psi(x) = x/|x|^2$ which is known ([1]) to be a proper biharmonic map.

Example 11. The map $\phi : (\mathbb{R}^2, g = e^{x_2}(dx_1^2 + dx_2^2)) \longrightarrow (\mathbb{R}^2 \times \mathbb{R}^3, e^{x_2}(dx_1^2 + dx_2^2) + dx_3^2 + dx_4^2 + dx_5^2)$ defined by $\psi(x_1, x_2) = (x_1, x_2, \cos x_1, \sin x_1, x_2)$ is a proper biharmonic map. This is because the map ϕ is the graph of the map $\psi : (\mathbb{R}^2, g = e^{x_2}(dx_1^2 + dx_2^2)) \longrightarrow \mathbb{R}^3$, $\psi(x_1, x_2) = (\cos x_1, \sin x_1, x_2)$, which is a proper biharmonic conformal immersion of $\mathbb{R}^2$ into Euclidean space $\mathbb{R}^3$ ([25]).

Another consequence of Theorem 3.3 is the following

Corollary 3.5. *Let $\psi : (M, g) \longrightarrow (N, h)$ be a map. Then,*
(1) *the graph $\phi : (M, \phi^*(g \times h)) \longrightarrow (M \times N, g \times h)$ with $\phi(x) = (x, \psi(x))$, is a biharmonic isometric embedding if and only if both maps $\varphi : (M, g + \psi^* h) \longrightarrow (M, g)$ with $\varphi(x) = x$ and $\psi : (M, g + \psi^* h) \longrightarrow (N, h)$ are biharmonic. Furthermore, if one of the component maps is proper biharmonic then so is the graph ϕ;*
(2) *A submanifold $\psi : (M, g) \longrightarrow (N, h)$ is biharmonic if and only if its graph $\phi : (M, g) \longrightarrow (M \times N, g \times h)$ with $\phi(x) = (x, \psi(x))$ is a biharmonic embedded submanifold.*
(3) *The graph of $\psi : (M^2, g) \longrightarrow (N, h)$ is a biharmonic isometric embedding if and only if the map $\psi : (M^2, g + \psi^* h) \longrightarrow (N, h)$ is biharmonic.*

Proof. The Statement (1) follows from Theorem 3.3 immediately. For Statement (2), note that if $\psi : (M, g) \longrightarrow (N, h)$ is an isometric immersion, then $\phi^*(g \times h) = 2g$, from which and Theorem 3.3 we obtain the required result. To prove Statement (3), first note that, by Theorem 3.3, the biharmonicity of the graph ϕ is equivalent to the biharmonicity of $\varphi : (M^2, g + \psi^* h) \longrightarrow (M^2, g)$ and $\psi : (M^2, g + \psi^* h) \longrightarrow (N, h)$. Since φ is an identity map between two Riemann surfaces, one can easily check that φ is harmonic. It follows that ϕ is biharmonic if and only if ψ is biharmonic. □

4. Biharmonic graphs and Chen's conjecture

Concerning biharmonic submanifolds of Euclidean space we have the following **Chen's Conjecture:** any biharmonic submanifold of Euclidean space is minimal.

Jiang [16], Chen-Ishikawa [8] proved that any biharmonic surface in $\mathbb{R}^3$ is minimal; Dimitrić [9] showed that any biharmonic curves in $\mathbb{R}^n$ is a part of a straight line, any biharmonic submanifold of finite type in $\mathbb{R}^n$ is minimal, any pseudo-umbilical submanifolds $M^m \subset \mathbb{R}^n$ with $m \neq 4$ is minimal, and any biharmonic hypersurface in $\mathbb{R}^n$ with at most two distinct principal curvatures is minimal; it is proved in [14] that any biharmonic hypersurface in $\mathbb{R}^4$ is minimal. However, the conjecture is still open.

As any hypersurface of Euclidean space is locally the graph of a real-valued function, we believe that the next strategic step in attacking Chen's conjecture is to check whether a biharmonic graph in Euclidean space is minimal. The following theorem gives conditions on a real-valued function whose graphs produce biharmonic hypersurfaces in a Euclidean space, from which we obtain an equivalent analytic formulation of Chen's conjecture.

Theorem 4.1. *(I) A function $f : \mathbb{R}^m \supseteq D \longrightarrow \mathbb{R}$ has biharmonic graph $\{(x, f(x)) : x \in D\} \subseteq \mathbb{R}^{m+1}$ if and only if of f is a solution of*

$$\begin{cases} \Delta^2 f = 0, \\ (\Delta f_k)\Delta f + 2g(\nabla f_k, \nabla \Delta f) = 0, \quad k = 1, 2, \ldots, m, \end{cases} \tag{21}$$

where the Laplacian Δ and the gradient ∇ are taken with respect to the induced metric $g_{ij} = \delta_{ij} + f_i f_j$.
(II) The following statements are equivalent:
(i) Any biharmonic hypersurface in Euclidean space $\mathbb{R}^{m+1}$ is minimal (Chen's Conjecture for biharmonic hypersurfaces [7]);
(ii) Any solution of Equation (21) is a harmonic function, i.e., $\Delta f = 0$.

Proof. To prove Statement (I), let $f : \mathbb{R}^m \supseteq D \longrightarrow \mathbb{R}$ with $f = f(x_1, \ldots, x_m)$ be a function. Then, by Corollary 3.5, the graph $\phi : \mathbb{R}^m \supseteq D \longrightarrow \mathbb{R}^{m+1}$ with $\phi(x) = (x_1, \ldots, x_m, f(x_1, \ldots, x_m))$ is a biharmonic hypersurface if and only if all component functions are biharmonic with respect to the induced metric $g = (g_{ij}) = (\delta_{ij} + f_i f_j)$. This is equivalent to f being a solution of

$$\begin{cases} \Delta^2 f = 0, \\ \Delta^2 x_k = 0, \quad k = 1, 2, \ldots, m. \end{cases} \tag{22}$$

To prove the equivalence of (21) and (22) we need to find a way to compute the Laplacian on the hypersurface. To this end, we use the standard Cartesian coordinates $(x_1, \ldots, x_m, x_0)$ in $\mathbb{R}^{m+1}$ and with respect to which the standard Euclidean metric is given by $h_0 = dx_1^2 + \ldots + dx_m^2 + dx_0^2$. We will use the notations $\partial_i = \frac{\partial}{\partial x_i}, i = 0, 1, 2, \ldots, m$, $f_i = \frac{\partial f}{\partial x_i}$, and $f_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}$.

Claim: The Laplacian on the hypersurface with respect to the induced metric g is given by

$$\Delta u = g^{ij} u_{ij} - \Delta f \sum_{i=1}^{m} f_i u_i, \tag{23}$$

where u is any function defined on $D \subseteq \mathbb{R}^m$ and $\Delta f = \frac{g^{ij} f_{ij}}{1 + |\nabla_0 f|^2}$.

Proof of the Claim: It is easy to check that

$$\begin{cases} X_i = \mathrm{d}\phi(\partial_i) = \partial_i + f_i \partial_0, \quad i = 1, 2, \ldots, m, \\ \xi = (-\nabla_0 f + \partial_0)/\sqrt{1 + |\nabla_0 f|^2}, \end{cases} \tag{24}$$

where ∇_0 denoting the Euclidean gradient, form a frame (not necessarily an orthonormal one) on $\mathbb{R}^{m+1}$ adapted to the hypersurface with ξ being the unit normal vector field. Clearly, the induced metric on the hypersurface has components

$$(g_{ij}) = (\langle X_i, X_j \rangle) = (\delta_{ij} + f_i f_j), \quad (g^{ij}) = (g_{ij})^{-1}. \tag{25}$$

Let $\bar{\nabla}$ denote the connection of the ambient Euclidean space. Then, we have

$$\bar{\nabla}_{X_i} X_j = \bar{\nabla}_{(f_i \partial_0 + \partial_i)} (f_j \partial_0 + \partial_j) = f_{ij} \partial_0. \tag{26}$$

The second fundamental form of the hypersurface with respect to the frame $\{X_i\}$ is give by

$$b_{ij} = b(X_i, X_j) = \langle \bar{\nabla}_{X_i} X_j, \xi \rangle = \frac{f_{ij}}{\sqrt{1 + |\nabla_0 f|^2}}. \tag{27}$$

We can apply the Gauss formula $\bar{\nabla}_{X_i} X_j - \nabla_{X_i} X_j = b(X_i, X_j)\xi$ for a hypersurface to have

$$\begin{aligned} \nabla_{X_i} X_j &= \bar{\nabla}_{X_i} X_j - b(X_i, X_j)\xi \\ &= f_{ij}(\nabla_0 f + |\nabla_0 f|^2 \partial_0)/(1 + |\nabla_0 f|^2), \end{aligned} \tag{28}$$

from which, together with the formula $\Delta u = g^{ij}(X_j X_i u - \nabla_{X_i} X_j \, u)$, we obtain the Claim.

Using (23) and a straightforward computation we have

$$\Delta f = \frac{g^{ij} f_{ij}}{1+|\nabla_0 f|^2}, \tag{29}$$

and

$$\Delta x_k = -f_k \Delta f, \quad k = 1, 2, \ldots, m. \tag{30}$$

A further computation using (23), (29) and (30) shows that (22) is indeed equivalent to (21), which ends the proof of the first statement.

For statement (II), first notice that the mean curvature of the graph is given by

$$mH = g^{ij} b_{ij} = \frac{g^{ij} f_{ij}}{\sqrt{1+|\nabla_0 f|^2}} = \sqrt{1+|\nabla_0 f|^2}\, \Delta f. \tag{31}$$

It follows that the function f has minimal graph if and only if the graph ϕ, as an isometric embedding, is harmonic. This is equivalent to the vanishing of both (29) and (30), which is equivalent to f being a harmonic function with respect to the metric $\phi^* h_0 = g_{ij} = \delta_{ij} + f_i f_j$ (and, by (31), it is equivalent to the mean curvature H vanishes identically).

By the implicit function theorem any hypersurface is locally the graph of a function. It follows that the existence of a non-harmonic biharmonic hypersurface is equivalent to the existence of a solution of (21) which is not harmonic, from which we obtain the Statement (II) which completes the proof of the theorem. $\square$

Remark 7. We notice that the well-known Bernstein's conjecture and Chen's conjecture on biharmonic hypersurfaces are similar in the following sense: Bernstein conjecture claimed that for $m \geq 2$, any entire solution of the minimal graph equation

$$\Delta f = 0 \qquad \Longleftrightarrow \qquad \sum_{i,j=1}^{m} \left(\delta_{ij} - \frac{f_i f_j}{1+|\nabla f|^2} \right) f_{ij} = 0,$$

is trivial, i.e., f is an affine function. According to our Theorem 4.1, Chen's conjecture for biharmonic hypersurfcaes is equivalent to claiming that for $m \geq 2$, any solution of the biharmonic graph equation

$$\begin{cases} \Delta(\Delta f) = 0, \\ (\Delta f_k)\Delta f + 2g(\nabla f_k, \nabla \Delta f) = 0, \quad k = 1, 2, \ldots, m, \end{cases} \tag{32}$$

is trivial, i.e., $\Delta f = 0$.

We know that Bernstein's conjecture is true for $m = 2$ (Bernstein, 1915), $m = 3$ (De Giorgi,1965), $m = 4$ (Almgren, 1966), and $m \leq 7$ (J. Simons, 1968); however, it fails to be true for $m > 7$ (Bombieri-De Giorgi-Giusti, 1969). We are not sure whether the similarity between the two conjectures implies that Chen's conjecture is more likely to be false. We do hope that the equivalent analytic formulation of Chen's conjecture for biharmonic hypersurfaces and its interesting link to Bernstein's conjecture will attract more work and especially more analytic study of the conjecture.

References

[1] P. Baird and D. Kamissoko, *On constructing biharmonic maps and metrics*, Ann. Global Anal. Geom. 23 (2003), no. 1, 65–75.

[2] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monogr. (N.S.) No. 29, Oxford Univ. Press (2003).

[3] A. Balmus, S. Montaldo, C. Oniciuc, *Biharmonic maps between warped product manifolds*, J. Geom. Phys. 57 (2007), no. 2, 449–466.

[4] A. Balmus, S. Montaldo and C. Oniciuc, *Properties of biharmonic submanifolds in spheres*, Preprint 2009, arXiv:0908.3063

[5] R. Caddeo, S. Montaldo, and C. Oniciuc, *Biharmonic submanifolds of* S^3, Internat. J. Math. 12 (2001), no. 8, 867–876.

[6] R. Caddeo, S. Montaldo and C. Oniciuc, *Biharmonic submanifolds in spheres*, Israel J. Math. 130 (2002), 109–123.

[7] B. Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math. 17 (1991), no. 2, 169–188.

[8] B. Y. Chen and S. Ishikawa, *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces*, Kyushu J. Math. 52 (1998), no. 1, 167–185.

[9] I. Dimitrić, *Submanifolds of* E^m *with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sinica 20 (1992), no. 1, 53–65.

[10] J. Eells, *Minimal graphs*, Manuscripta Math. 28 (1979), no. 1-3, 101–108.

[11] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. 10 (1978), no. 1, 1–68.

[12] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 1964 109–160.

[13] J. Eells and J. C. Wood, *The existence and construction of certain harmonic maps*, Symposia Mathematica, Vol. XXVI (Rome, 1980), pp. 123–138, Academic Press, London-New York, 1982.

[14] T. Hasanis and T. Vlachos, *Hypersurfaces in* E^4 *with harmonic mean curvature vector field*, Math. Nachr. 172 (1995), 145–169.

[15] G. Y. Jiang, 2-*Harmonic maps and their first and second variational formulas*, Chin. Ann. Math. Ser. A 7(1986) 389-402.

[16] G. Y. Jiang, *Some non-existence theorems of* 2-*harmonic isometric immersions into Euclidean spaces* , Chin. Ann. Math. Ser. 8A (1987) 376-383.

[17] G. Y. Jiang, 2-*harmonic isometric immersions between Riemannian manifolds*. Chinese Ann. Math. Ser. A 7 (1986), no. 2, 130–144.

[18] E. Loubeau and C. Oniciuc, *On the biharmonic and harmonic indices of the Hopf map*, Trans. Amer. Math. Soc. 359 (2007), no. 11, 5239–5256.

[19] E. Loubeau and C. Oniciuc, *The index of biharmonic maps in spheres*, Comp. Math. 141 (2005), 729-745.

[20] E. Loubeau and Y. -L. Ou, *Biharmonic maps and morphisms from conformal mappings*, Tôhoku Math J., 62 (1), 2010, to appear.

[21] Y.-L. Ou. *On constructions of harmonic morphisms into Euclidean spaces*, J. Guangxi University for Nationalities, 1996 (1).

[22] Y. -L. Ou, *Complete lifts of harmonic maps and morphisms between Euclidean spaces*, Beiträge Algebra Geom. 37 (1996), no. 1, 31–40.

[23] Y.-L. Ou, *Biharmonic morphisms between Riemannian manifolds*, Geometry and topology of submanifolds, X (Beijing/Berlin, 1999), 231–239.

[24] Y. -L. Ou, *p-Harmonic morphisms, biharmonic morphisms, and nonharmonic biharmonic maps*, J. Geom. Phys. 56(2006) 358-374.

[25] Y. -L. Ou, *On conformal biharmonic immersions*, Ann. Global Analysis and Geometry, 36(2) (2009), 133-142.

[26] Y. -L. Ou, *Biharmonic hypersurfaces in Riemannian manifolds*, Preprint, 2009, arXiv:0901.1507.

[27] S. Ouakkas, *Biharmonic maps, conformal deformations and the Hopf maps*, Differential Geom. Appl. 26 (2008), no. 5, 495–502.

[28] M. Parker, *Orthogonal multiplications in small dimensions*, Bull. London Math. Soc. 15 (1983), no. 4, 368–372.

Department of Mathematics,
Texas A & M University-Commerce,
Commerce TX 75429,
USA.
E-mail:yelin_ou@tamu-commerce.edu